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A Comment on the Roe-Woodroofe Construction of Poisson Confidence Intervals

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Abstract

We consider the Roe-Woodroofe construction of confidence intervals for the case of a Poisson distributed variate where the mean is the sum of a known background and an unknown non-negative signal. We point out that the intervals do not have coverage in the usual sense but can be made to have such with a modification that does not affect the believability and other desirable features of this attractive construction. A similar modification can be used to provide coverage to the construction recently proposed by Cousins for the Gaussian-with-boundary problem.

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I. INTRODUCTION

A problem of long-standing interest is that of setting confidence intervals for an unknown non-negative signal μ in the presence of a known mean background b when the measurement n is Poisson distributed as $p(n;\mu+b)$. When $n<b$, the usual estimate for μ, i.e. $n-b$, is negative, leading in most constructions to small upper limits that imply unrealistically high confidence in small values of μ. In a recent paper, Roe and Woodroofe [1] propose a construction that produces more believable intervals and contains the unifying feature that one need not decide beforehand whether to set a confidence interval or an upper confidence bound. However, since the Roe-Woodroofe confidence belt (of confidence level α) is not constructed from an unconditional probability density and does not have coverage in the usual sense (i.e. unconditional coverage), one cannot state that the unconditional probability of the interval enclosing the true value is at least α. Our comment is that a straightforward modification of the Roe-Woodroofe confidence belt gives it coverage, making the construction effectively an *ordering principle* applied to the Poisson pdf, albeit reached by circuitous means.

II. ROE-WOODROOFE CONFIDENCE INTERVALS

Roe and Woodroofe are motivated by the observation [1] that the measurement $n=0$ implies that zero signal (as well as zero background) is seen; thus, the resulting estimate for μ is zero, independent of b. They argue therefore that the confidence interval for μ for n=0 must be independent of b. Extending the argument, they note that for any observation n, one has observed a result n from the Poisson pdf $p(n;\mu+b)$ *and* a background of at most n. They formulate a method of obtaining confidence intervals based on the conditional probability to observe n given a background $\leq n$ and obtain the desired result for $n=0$

and approximately the classical confidence intervals for $n > b$. While they identify their method as an ordering principle, it is not one in the same sense as Ref.s [2] and [3] which explicitly choose a confidence belt of probability α using the Poisson pdf $p(n; \mu + b)$ and the Likelihood Ratio Construction and invert it to find confidence intervals. The latter methods do not obtain intervals that are independent of b for $n = 0$ and yield confidence intervals which are unphysically small for $n < b$.

Although the Roe-Woodroofe construction does not have coverage in the usual sense , it can be easily modified to obtain coverage, by retaining the left-hand boundary of the confidence belt and adjusting the right-hand boundary so that for all μ the horizontal intervals contain probability α. In Fig. 1 we show the Roe-Woodroofe 90% intervals for $b = 3$ along with one-sided and central confidence belts * for the Poisson distribution without background. We note that the Roe-Woodroofe horizontal intervals do not coincide with the one-sided intervals shown for $\mu < 2.44$. Therefore for some values of μ in this range, the confidence belt does not satisfy the coverage requirement that $\geq$90% of the probability is contained. Because coverage cannot be exact when the variable is discrete, the *error* for the example given here is not of great numerical significance. The minimum coverage of

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*We show the confidence belt consisting of central intervals $[n_1(\mu_0),\ n_2(\mu_0)]$ containing at least 90% of the probability for unknown Poisson mean μ_0 in the absence of any known background (dotted) and the 90% one-sided belt consisting of intervals $[0,\ n_{os}(\mu_0)]$(dashed). There is some arbitrariness in the choice of a central interval for a discrete variate. We choose the smallest interval such that there is $\geq$ 90% of the probability in the center and $\leq$5%, but as close as possible to 5%, on the right. The alternative of requiring $\leq$5%, but as close as possible to 5%, on the left gives slightly less symmetrical intervals. For the latter choice the 90% Poisson upper limit for $n = 0$ is $\mu_0 = 3.0$ compared to $\mu_0 = 2.62$ for our choice. For $\mu_0 < 2.62$, according to this prescription, one cannot construct an interval containing probability $>$ 90% that does not include $n = 0$ and we adopt 90% one-sided intervals.

~ 0.87 is obtained at $\mu \sim 0.4$. Undercoverage is more severe for greater b; for $b = 10.0$, the minimum coverage is ~ 0.78. However, it is desirable to have coverage, which we obtain as shown in Fig. 2 where we have changed the right side of the confidence belt so that the horizontal intervals contain probability $\geq$90%. We note that the confidence intervals for small n, i.e. $n < b$, are unchanged. Intervals for both constructions are given in Table I.

It would be nice to devise an ordering principle that can be directly applied to the Poisson pdf $p(n; \mu + b)$ to obtain the confidence belt shown in Fig. 2, if only because the construction we have used here is aesthetically unpleasing. This method, which consists of first determining vertical intervals per Ref. [1], and then fixing them, leaves something to be desired. However, in the end the method of construction does not really matter. What results here is an ordering procedure that yields a confidence belt with coverage and produces physically sensible intervals.

B. Roe has noted [4] that our modification is equally applicable to a construction due to R. Cousins, in which the Roe-Woodroofe method of conditioning is applied to the Gaussian-with-boundary [5] problem. Here, for example, an interval of confidence level α is sought for an unknown non-negative signal μ and the measurements x are normally distributed as N(x;μ). As for the Roe-Woodroofe construction referred to above, the Cousins construction produces physically sensible confidence intervals for all x including $x < 0$. However this construction significantly undercovers for $\mu < 0.5$ and significantly overcovers for $\mu \sim 1$. In order to produce exact coverage using the Cousins construction, we retain the left hand (upper) curve of the confidence belt $x_l(\mu)$ and recalculate the right hand (lower) curve $x_r(\mu)$ so that the horizontal intervals contain probability α using:

$$2\alpha = erf(\frac{\mu - x_l}{\sqrt{2}}) + erf(\frac{x_r - \mu}{\sqrt{2}}). \tag{1}$$

III. CONCLUSION

For the case of Poisson distributed measurements n with a non-negative signal mean μ and known mean background b, the Roe-Woodroofe construction produces well-behaved confidence intervals, particularly for $n < b$ where other constructions yield unphysically small intervals. Since the construction is not based on integrating probabilities that arise from an unconditional pdf, it does not produce a confidence belt with coverage in the usual frequentist sense. We suggest a modification that provides coverage while preserving the desirable features of the construction. While the changes introduced by this modification are relatively small for the example given here (they are larger for greater b), nevertheless the procedure corrects a formal defect in the original construction. A similar modification provides coverage for a construction recently discussed by R. Cousins for the Gaussian-with-boundary problem.

REFERENCES

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[2] G. J. Feldman and R. D. Cousins, Phys. Rev. D**57**, 3873 (1998).

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[4] B. P. Roe, private communication.

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TABLES

n(observed)	Roe-Woodroofe		Modified	
	Lower	Upper	Lower	Upper
0	0.0	2.44	0.0	2.44
1	0.0	2.95	0.0	2.95
2	0.0	3.75	0.0	3.75
3	0.0	4.80	0.0	4.80
4	0.0	6.01	0.0	6.01
5	0.0	7.28	0.0	7.28
6	0.42	8.40	0.16	8.42
7	0.96	9.58	0.90	9.58
8	1.52	10.99	1.66	11.02
9	1.88	12.23	2.44	12.23
10	2.64	13.50	2.98	13.51
11	3.04	14.80	3.75	14.77
12	4.01	15.90	4.52	16.01

TABLE I. Comparison of confidence intervals for the Roe-Woodroofe and modified Roe-Woodroofe constructions

FIGURES

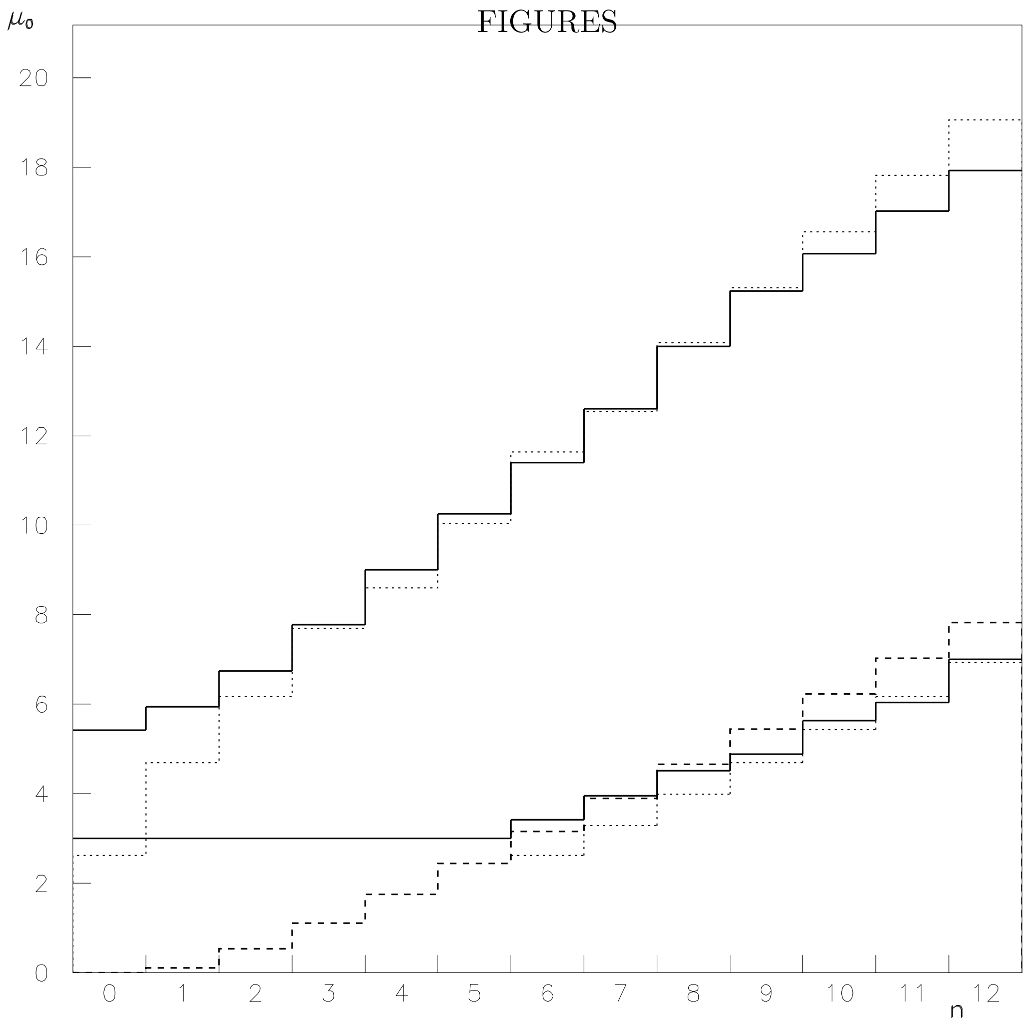

FIG. 1. 90% Poisson confidence belts for unknown non-negative signal μ in the presence of a background with known mean b taken to be 3.0, where n is the result of a single observation. The solid belt is the Roe-Woodroofe construction, the dotted belt the central construction and the dashed belt the one-sided construction of 90% Poisson lower limits. Here $\mu_0 = \mu+b$ is the parameter representing the mean of signal plus background. We illustrate confidence belts in this manner to demonstrate the absence of coverage for the Roe-Woodroofe construction and to emphasize that a naive approach to setting a confidence interval for μ leads to a null interval for sufficiently small $n < b$, in this case $n = 0$. The solid line Roe-Woodroofe lower limit for $n \leq 5$ is at $\mu = 0$.

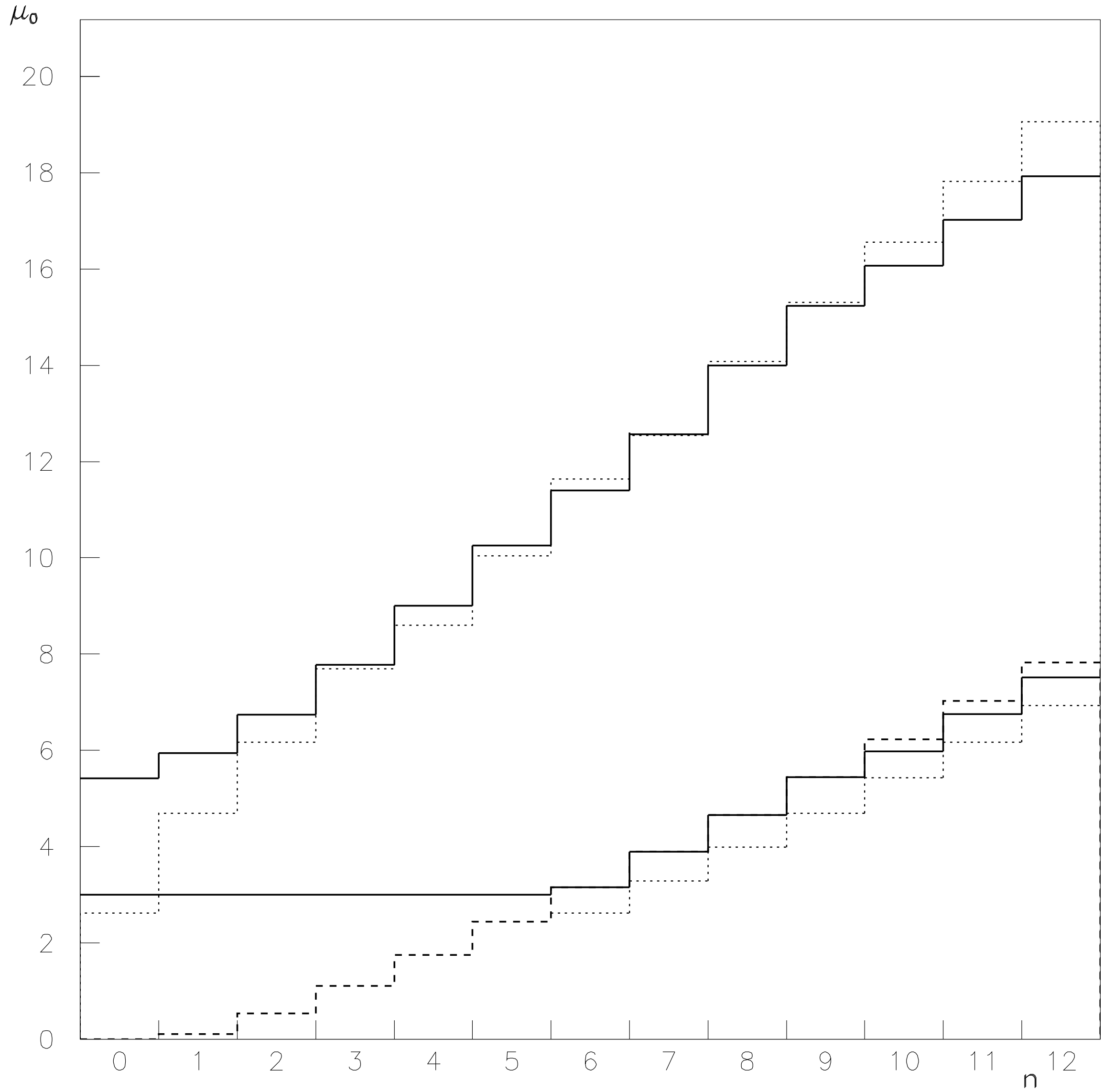

FIG. 2. 90% Poisson confidence belts described in Fig. 1 where the solid belt is modified as described in the text to give coverage. The dotted and dashed belts are described in the Fig. 1 caption. For $n = 6, 7, 8, 9$ the lower limits of the confidence intervals coincide with the one-sided 90% Poisson lower limits. This guarantees $\geq$90% probability within the horizontal intervals.